Available online at www.sciencedirect.com
SciVerse ScienceDirect
Advances in Mathematics 228 (2011) 3300–3319
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Constructing infra-nilmanifolds admitting an Anosov diffeomorphism

Karel Dekimpe *, Kelly Verheyen [1]

K.U.Leuven Campus Kortrijk, Universitaire campus, B-8500 Kortrijk, Belgium

Received 10 June 2010; accepted 19 August 2011
Available online 7 September 2011
Communicated by R. Daniel Mauldin

Abstract

In this paper we establish an algebraic characterization of those infra-nilmanifolds modeled on a free c-step nilpotent Lie group and with an abelian holonomy group admitting an Anosov diffeomorphism. We also develop a new method for constructing examples of infra-nilmanifolds having an Anosov diffeomorphism.

Keywords: Infra-nilmanifold; Anosov diffeomorphism; Free nilpotent Lie group

1. Preliminaries

In the last decade, there has been a growing interest in the study of Anosov diffeomorphisms (e.g. [2,3,5,11,12,14–16,19]) on manifolds. However, all of these papers are mainly dealing with the class of nilmanifolds and up till now, except from the work of H. Porteous [20] on flat manifolds (of which we will describe the main result in Section 1.2), very little is known about Anosov diffeomorphisms on the more general class of infra-nilmanifolds. In [7] the first steps in this direction were set and we continue this line of research in this paper.

* Corresponding author.
E-mail address: Karel.Dekimpe@kuleuven-kortrijk.be (K. Dekimpe).
[1] The second author was supported by a PhD fellowship of the Research Foundation – Flanders (FWO). Research supported by the research Fund of the Katholieke Universiteit Leuven.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2011.08.008

In this first section we start by recalling some basic material about infra-nilmanifolds and Anosov diffeomorphisms on them. Thereafter we recall the notion of a rational holonomy representation as introduced in [7].

Based on this rational holonomy representation, we will establish a criterium for the existence of an Anosov diffeomorphism for a specific class of infra-nilmanifolds (Sections 2 and 3).

In Sections 4 and 5 we develop a method for constructing infra-nilmanifolds with a given rational holonomy representation, allowing us to provide interesting examples of infra-nilmanifolds allowing Anosov diffeomorphisms (we give such an example in Section 6).

1.1. Anosov diffeomorphisms on infra-nilmanifolds

In this section we briefly recall the notion of an infra-nilmanifold (for more information, the reader can consult [4,6,13]).

Let L be a connected and simply connected nilpotent Lie group, then $\mathrm{Aff}(L) = L \rtimes \mathrm{Aut}(L)$ is called the affine group of L and it acts on L via $\forall (l, \varphi) \in \mathrm{Aff}(L)$, $\forall x \in L$: $(l, \varphi) \cdot x = l\varphi(x)$.

Now, fix a compact subgroup C of $\mathrm{Aut}(L)$. A uniform and discrete subgroup E of $L \rtimes C \subseteq \mathrm{Aff}(L)$ is called an **almost-crystallographic group**. If moreover E is torsion-free, it is called an **almost-Bieberbach group**. In the case where L is $\mathbb{R}^n$, we say that E is a **crystallographic group**, resp. a **Bieberbach group**.

Such an almost-Bieberbach group E acts properly discontinuously on L, and we say that $M = E \setminus L$ is an **infra-nilmanifold** modeled on the nilpotent Lie group L. When $L = \mathbb{R}^n$ (or E is a Bieberbach group), the manifold $E \setminus \mathbb{R}^n$ is a flat manifold. Note that the fundamental group of M is E. Every almost-crystallographic group fits in a short exact sequence

$$1 \to N \to E \to F \to 1, \tag{1}$$

in which F is the (finite) **holonomy group** of E (or of the infra-nilmanifold $E \setminus L$), and $N = E \cap L$ is a finitely generated torsion-free nilpotent group that is maximal nilpotent in E. Moreover, N will be a uniform lattice of L.

Conversely, any group E fitting in such an extension (1) in which F is finite and N is maximal nilpotent in E, can be realized as an almost-crystallographic group. The connected and simply connected nilpotent Lie group L needed for this realization is uniquely determined and is in fact the Malcev completion $N_{\mathbb{R}}$ of N (i.e. the unique connected and simply connected nilpotent Lie group L containing N as a uniform lattice).

If $E \subseteq \mathrm{Aff}(L)$ is an almost-Bieberbach group and $f = (l, \varphi) \in \mathrm{Aff}(L)$ such that $fEf^{-1} = E$, then the map

$$f : L \to L : x \mapsto l\varphi(x)$$

induces a diffeomorphism on $E \setminus L$, which is called an **affine diffeomorphism** of the infra-nilmanifold $E \setminus L$. If moreover φ is hyperbolic, then the induced morphism is said to be a **hyperbolic infra-nilmanifold automorphism**. These hyperbolic infra-nilmanifold automorphisms are prototype examples of Anosov diffeomorphisms, which are diffeomorphisms on a manifold M for which the tangent bundle TM continuously splits into an expanding part and a contracting part. To be more precise, a diffeomorphism $f : M \to M$ is an **Anosov diffeomorphism** if and only if there exist a continuous splitting $TM = E^u \oplus E^s$ and real constants $c > 1$ and $0 < \lambda < 1$ such that for some Riemannian metric $\| \ \|$ on TM:

$$\forall n > 0: \quad \left\| df^n(v) \right\| \leqslant c\lambda^n \|v\| \quad \text{for } v \in E^s \quad \text{and} \quad \left\| df^n(v) \right\| \geqslant c\lambda^{-n} \|v\| \quad \text{for } v \in E^u.$$

Moreover, there is a result of Manning [17] saying that essentially all Anosov diffeomorphisms on an infra-nilmanifold are of the above type:

Theorem *(Manning, 1974). Any Anosov diffeomorphism f of an infra-nilmanifold M is topologically conjugate to a hyperbolic infra-nilmanifold automorphism.*

Hence, an infra-nilmanifold admits an Anosov diffeomorphism if and only if it admits a hyperbolic infra-nilmanifold automorphism.

1.2. The rational holonomy representation

In this section we recall the notion of the rational holonomy representation of a given almost-crystallographic group (see [7]).

We consider an almost-crystallographic group E fitting in a short exact sequence (1) and we denote by $N_{\mathbb{Q}}$ the radicable hull (or rational Malcev completion, see e.g. [21]) of N. Using the fact that every automorphism of N has a unique lift to an automorphism of $N_{\mathbb{Q}}$, we can construct the following commutative diagram of groups:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, tail] & E \arrow[r] \arrow[d, tail, "i_1"'] & F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N_{\mathbb{Q}} \arrow[r] & E_{\mathbb{Q}} \arrow[r] & F \arrow[r] & 1,
\end{tikzcd}$$

in which the bottom extension splits. So we can fix a splitting morphism $s : F \to E_{\mathbb{Q}}$ and we use $\varphi : F \to \mathrm{Aut}(N_{\mathbb{Q}})$ to denote the induced morphism (where $\varphi(f)(n) = s(f)ns(f)^{-1}$ for all $n \in N_{\mathbb{Q}}$). We will refer to the map φ as being the **rational holonomy representation** determined by E. We remark here that the rational holonomy representation does depend upon the choice of the chosen splitting morphism, but this will not play a role in what follows. In fact, we will mainly use the induced **abelianized rational holonomy representation**

$$\bar{\varphi} : F \to \mathrm{Aut}\left(\frac{N_{\mathbb{Q}}}{[N_{\mathbb{Q}}, N_{\mathbb{Q}}]}\right)$$

defined by

$$\bar{\varphi}(f)\bigl(n[N_{\mathbb{Q}}, N_{\mathbb{Q}}]\bigr) = \varphi(f)(n)[N_{\mathbb{Q}}, N_{\mathbb{Q}}],$$

which is independent of the chosen s.

We can (and will) identify $\mathrm{Aut}(\frac{N_{\mathbb{Q}}}{[N_{\mathbb{Q}}, N_{\mathbb{Q}}]})$ with $\mathrm{GL}(n, \mathbb{Q})$, for some n.

Using the rational holonomy representation, H. Porteous gives a criterion of when a flat manifold does admit an Anosov diffeomorphism.

Theorem 1.1. *(See H. Porteous [20, Theorem 6.1].) Let $M = E \setminus \mathbb{R}^n$ be a flat manifold with associated Bieberbach group E and rational holonomy representation $\varphi : F \to \mathrm{GL}(n, \mathbb{Q})$. Then*

$$M \text{ admits an Anosov diffeomorphism}$$
$$\Updownarrow$$
$$\text{Each irreducible component of } \varphi \text{ which occurs with multiplicity one is reducible over } \mathbb{R}.$$

To end this section, we introduce the following notions:

Definition 1.2. We say that $A \in \mathrm{GL}(n, \mathbb{Q})$ is c**-hyperbolic** if its eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ satisfy the following condition:

$$\forall r \in \{1, \ldots, c\},\ \forall i_1, i_2, \ldots, i_r \in \{1, \ldots, n\}: \quad |\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}| \neq 1.$$

We say that $A \in \mathrm{GL}(n, \mathbb{Q})$ is **integral** if its characteristic polynomial $\varphi_A(X)$ has integer coefficients and constant term ± 1.

Analogously, we say that an automorphism of $\mathbb{Q}^n$ is integral, resp. c-hyperbolic, if its matrix representation (with respect to an arbitrary basis of $\mathbb{Q}^n$) is integral, resp. c-hyperbolic. Remark that, given an integral automorphism f of $\mathbb{Q}^n$, we can always find a basis of $\mathbb{Q}^n$ such that, with respect to this basis, the matrix representation of f is in $\mathrm{GL}(n, \mathbb{Z})$.

In [8] (see also [2]), we proved the following:

Theorem 1.3. *For any positive integer $n > 1$, there exists an $(n-1)$-hyperbolic matrix $A \in \mathrm{GL}(n, \mathbb{Z})$.*

As a consequence of this result S.G. Dani ([2], see also [8]) proved that a nilmanifold modeled on a free c-step nilpotent Lie group with k generators admits an Anosov diffeomorphism if and only if $k < c$. In this paper we will generalize this result and study the more general class of infra-nilmanifolds modeled on such free nilpotent Lie groups.

2. Classification

In this section we will establish a criterion to decide on the existence of an Anosov diffeomorphism for certain infra-nilmanifolds modeled on a free nilpotent Lie group. Recall that a Lie group is said to be free nilpotent (of class c, on k generators) if its Lie algebra is free nilpotent (of class c, on k generators).

As we proved in [7, Theorem B], we already have the following algebraic characterization for infra-nilmanifolds admitting an Anosov diffeomorphism:

Theorem 2.1. *Let M be an infra-nilmanifold modeled on a free c-step nilpotent Lie group, and denote with $\bar{\varphi} : F \to \mathrm{Aut}(\frac{N_\mathbb{Q}}{[N_\mathbb{Q}, N_\mathbb{Q}]})$ its abelianized rational holonomy representation. Then*

$$M \text{ admits an Anosov diffeomorphism}$$
$$\Updownarrow$$
$$\text{There exists an integral, } c\text{-hyperbolic automorphism } \bar{\psi} \in \mathrm{Aut}\Big(\frac{N_\mathbb{Q}}{[N_\mathbb{Q}, N_\mathbb{Q}]}\Big)$$
$$\text{that commutes with every element of } \bar{\varphi}(F).$$

Proceeding in the same way as in [7, Theorem 6.1], we can translate the condition in the previous theorem as follows:

Theorem 2.2. *Let $T : F \to \mathrm{GL}(n, \mathbb{Q})$ be a representation of a finite group F and write $\Phi = \mathrm{Im}(T)$. Then the following holds*:

There exists an integral, c-hyperbolic matrix $C \in \mathrm{GL}(n, \mathbb{Q})$ commuting with every element of Φ

$$\Updownarrow$$

For every $\mathbb{Q}$-irreducible component $T_i : F \to \mathrm{GL}(n_i, \mathbb{Q})$ of T that occurs with multiplicity $m_i \leqslant c$, there exists an integral, c-hyperbolic matrix $C_i \in \mathrm{GL}(m_i n_i, \mathbb{Q})$ that commutes with every element of $\mathrm{Im}(\bigoplus_{j=1}^{m_i} T_i)$.

The previous theorem inspires us to have a closer look at $\mathbb{Q}$-irreducible representations of finite, abelian groups. If $T : F \to \mathrm{GL}(n, \mathbb{Q})$ is such a $\mathbb{Q}$-irreducible representation of a finite abelian group F, then $\mathrm{Im}(T)$ is cyclic. A generator M of $\mathrm{Im}(T)$ has a cyclotomic polynomial $\phi_d(X)$ as its characteristic polynomial, with d such that $\varphi(d) = n$ (where φ is the Euler function). The eigenvalues of M are exactly the n primitive dth roots of unity. Furthermore, if we denote $K = \mathbb{Q}(\xi_d)$ and $\mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_n\}$, then K is the splitting field of M, and we can find $v \in K^n$ such that $\{\sigma_1(v), \ldots, \sigma_n(v)\}$ is a basis of eigenvectors of M (with corresponding eigenvalues $\sigma_i(\xi_d)$). Furthermore, if we have $C \in \mathrm{GL}(n, \mathbb{Q})$ commuting with M, then M and C are simultaneously diagonalizable and $\mathrm{Spec}(C) = \{\sigma_1(\lambda), \ldots, \sigma_n(\lambda)\}$ (with corresponding eigenvectors $\sigma_i(v)$) for some $\lambda \in K$.

Theorem 2.3. *Let $T_1 : F \to \mathrm{GL}(n, \mathbb{Q})$ be a $\mathbb{Q}$-irreducible representation of a finite, abelian group F, c and m non-zero natural numbers, and write $T = \bigoplus_{j=1}^{m} T_1 : F \to \mathrm{GL}(mn, \mathbb{Q})$ to be the representation that is the direct sum of m times T_1. Then the following holds*:

There exists an integral, c-hyperbolic matrix $C \in \mathrm{GL}(mn, \mathbb{Q})$ that commutes with every element of $\mathrm{Im}(T)$

$$\Updownarrow$$

T_1 splits in more then $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.

Proof. Let $M = \begin{pmatrix} M_1 & 0_n & \ldots & 0_n \\ 0_n & M_1 & \ldots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & \ldots & M_1 \end{pmatrix}$ be a generator of $\mathrm{Im}(T)$, with $M_1 \in \mathrm{GL}(n, \mathbb{Q})$ having characteristic polynomial $\phi_d(X)$ and splitting field $K = \mathbb{Q}(\xi_d)$ (and $\varphi(d) = n$). Write $\mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_n\}$; we can find $P_1 \in \mathrm{GL}(n, K)$ such that

$$P_1^{-1} M_1 P_1 = \begin{pmatrix} \sigma_1(\xi_d) & 0 & \ldots & 0 \\ 0 & \sigma_2(\xi_d) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_n(\xi_d) \end{pmatrix}.$$

Remark that, in the case $n = 1$, T is $\pm$ the unit representation of degree m, and the theorem follows immediately since there exists an $(m-1)$-hyperbolic (and no m-hyperbolic) matrix $A \in \mathrm{GL}(m, \mathbb{Z})$ (Theorem 1.3).

So we may assume that $n \neq 1$. In this case, n is even and T_1 splits in exactly $\frac{n}{2}$ components over $\mathbb{R}$, so the second condition in the theorem is now equivalent to the condition $n > \frac{2c}{m}$.

First, suppose $n > \frac{2c}{m}$, and choose a field extension $\mathbb{Q} \subseteq K \subseteq L$ such that $[L : K] = m$ and L is Galois over $\mathbb{Q}$ (and hence also over K) (this can be done, e.g. see [18]). Without loss of generality, we may write $\mathrm{Gal}(L/K) = \{s_1, \ldots, s_m\}$ and $\mathrm{Gal}(L/\mathbb{Q}) = \{\eta_1, \eta_2, \ldots, \eta_{mn}\}$, in which $\eta_i = s_i$ for $1 \leqslant i \leqslant m$.

Since we know that $mn > 2c$, by Dirichlet's Units Theorem (using a generalization of [8, Lemma 2.4]) we can choose an algebraic unit μ in L such that

$$\forall k \leqslant c,\ \forall j_1, \ldots, j_k \in \{1, 2, \ldots, mn\}: \quad \left|\eta_{j_1}(\mu) \ldots \eta_{j_k}(\mu)\right| \neq 1.$$

If we take

$$f(X) = \prod_{i=1}^{m} \bigl(X - \eta_i(\mu)\bigr),$$

then the coefficients of f are fixed under $\mathrm{Gal}(L/K)$, and so $f(X) \in K[X]$. By construction, the roots of f are exactly $\eta_1(\mu), \ldots, \eta_m(\mu)$.

Consider the polynomials

$$m_i(X) = \sigma_i\bigl(f(X)\bigr), \quad 1 \leqslant i \leqslant n.$$

Then

$$m(X) = \prod_{i=1}^{n} m_i(X) \in \mathbb{Q}[X]$$

has coefficients in $\mathbb{Q}$ and $\eta_1(\mu)$ is a root of this polynomial, so all Galois-conjugates $\eta_i(\mu)$ are also roots. This means that $m(X)$ is (a power of) the minimal polynomial of μ over $\mathbb{Q}$; but since μ is an algebraic integer, this is a polynomial over $\mathbb{Z}$.

We may suppose that $\eta_{(j-1)m+1}(\mu), \eta_{(j-1)m+2} \ldots, \eta_{jm}(\mu)$ are the roots of $m_j(X)$ for every $1 \leqslant j \leqslant n$. Choose

$$\tilde{C} = \begin{pmatrix} 0 & 0 & \ldots & 0 & -C_0 \\ \mathbb{1}_n & 0 & \ldots & 0 & -C_1 \\ 0 & \mathbb{1}_n & \ldots & 0 & -C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbb{1}_n & -C_{m-1} \end{pmatrix}$$

where

$$C_i = \begin{pmatrix} c_{i,1} & 0 & \ldots & 0 \\ 0 & c_{i,2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_{i,n} \end{pmatrix}$$

in which $c_{i,j}$ is the coefficient corresponding to X^i in the polynomial $m_j(X)$.

Since the characteristic polynomial of $\tilde{C}$ is exactly $m(X)$, it is clear that $\tilde{C}$ is an integral and c-hyperbolic matrix. Moreover we see that $\tilde{C}$ commutes with $P^{-1}MP$, in which

$$P = \begin{pmatrix} P_1 & 0 & \dots & 0 \\ 0 & P_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_1 \end{pmatrix} \in \mathrm{GL}(mn, K).$$

Now

$$C = P\tilde{C}P^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & P_1C_0P_1^{-1} \\ \mathbb{1}_n & 0 & \dots & 0 & P_1C_1P_1^{-1} \\ 0 & \mathbb{1}_n & \dots & 0 & P_1C_2P_1^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbb{1}_n & P_1C_{m-1}P_1^{-1} \end{pmatrix}$$

will still be an integral, c-hyperbolic matrix, and it commutes with M. In the same way as in the proof of [7, Theorem 6.6], we can now show that $C \in \mathbb{Q}^{mn\times mn}$.

Conversely, suppose $n \leqslant \frac{2c}{m}$ $(n \neq 1)$, and suppose that $C \in \mathrm{GL}(mn, \mathbb{Q})$ is an integral matrix that commutes with M. We will show that C cannot be c-hyperbolic.

First suppose C has no multiple eigenvalues. Then since C and M commute, they will diagonalize simultaneously. This means we can find a field extension L of $\mathbb{Q}$ of degree mn, that contains the splitting field of both C and M and over which both C and M diagonalize. Since the splitting field of M is imaginary, L will be a totally imaginary field extension of $\mathbb{Q}$. The eigenvalues of C will hence be algebraic integers in a totally imaginary field extension of $\mathbb{Q}$ of degree mn, but since $mn \leqslant 2c$, they cannot satisfy the conditions of c-hyperbolicity.

Now suppose C has at least one multiple eigenvalue. Then its characteristic polynomial splits over $\mathbb{Q}$, hence over $\mathbb{Z}$, and we can write $\varphi_C(X) = p(X)q(X)$ in which $p, q \in \mathbb{Z}[X]$ with unit constant term (cf. [22, p. 19]). But then one of these two factors, say $p(X)$, has degree at most $\frac{mn}{2} \leqslant c$, and so the roots of $p(X)$ (which are a subset of the set of eigenvalues of C) can clearly not satisfy the conditions of c-hyperbolicity. $\square$

As a summary we find:

Theorem 2.4. *Let M be an infra-nilmanifold modeled on a free c-step nilpotent Lie group, with abelian holonomy group F and associated abelianized rational holonomy representation $\bar{\varphi} : F \to \mathrm{Aut}(\frac{N_{\mathbb{Q}}}{[N_{\mathbb{Q}}, N_{\mathbb{Q}}]})$. Then*

M admits an Anosov diffeomorphism

$$\Updownarrow$$

Each $\mathbb{Q}$-irreducible component of $\bar{\varphi}$ that occurs with multiplicity m, splits in more then $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.

3. Infra-nilmanifolds modeled on a free c-step nilpotent Lie group with few generators

Although we concentrated on infra-nilmanifolds with an abelian holonomy group thus far, it turns out that in case the infra-nilmanifold is modeled on a free c-step nilpotent Lie group with few generators, this is not really a restriction.

To be able to make this statement precise, we need some lemma's.

Lemma 3.1. *Suppose we have an integral and c-hyperbolic matrix $A \in \mathrm{GL}(n, \mathbb{Q})$ with $n \leqslant 2c+1$. Then $\varphi_A(X)$, the characteristic polynomial of A, is irreducible over $\mathbb{Q}$ and has no multiple roots.*

Proof. Let $A \in \mathrm{GL}(n, \mathbb{Q})$ be as described and suppose $\varphi_A(X)$ is reducible over $\mathbb{Q}$. Then $\varphi_A(X) = p(X)q(X)$, in which $p(X)$ and $q(X)$ are monic polynomials of degree at least one. Since $\varphi_A(X)$ has integer coefficients and unit constant term, the same will hold for $p(X)$ and $q(X)$.

Since $n \leqslant 2c + 1$, at least one of the polynomials $p(X)$ or $q(X)$ has degree $l \leqslant c$, suppose $p(X)$. Denote with $\{\lambda_1, \lambda_2, \dots, \lambda_l\}$ the roots of $p(X)$ (in which some λ_i may be the same). Then $\{\lambda_1, \dots, \lambda_l\} \subseteq \mathrm{Spec}(A)$, but $\lambda_1\lambda_2\dots\lambda_l = \pm 1$, which contradicts the c-hyperbolicity of A.

Remark that, as a direct consequence of the $\mathbb{Q}$-irreducibility, the roots of $\varphi_A(X)$ all have multiplicity one. $\square$

Lemma 3.2. *Let $\varphi : F \to \mathrm{GL}(n, \mathbb{Q})$ be a faithful representation of a finite group F. If A is an integral c-hyperbolic matrix with $n \leqslant 2c + 1$ and such that*

$$\forall f \in F: \quad A\varphi(f) = \varphi(f)A,$$

then F is an abelian group.

Proof. Consider the representation

$$\psi = i \circ \varphi : F \xrightarrow{\varphi} \mathrm{GL}(n, \mathbb{Q}) \overset{i}{\hookrightarrow} \mathrm{GL}(n, \mathbb{C})$$

and decompose $\psi = \psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_l$ into a direct sum of $\mathbb{C}$-irreducible representations. If every ψ_i has dimension one, then $\varphi(F)$ is diagonalizable over $\mathbb{C}$, meaning all elements of $\varphi(F)$ commute and F is abelian. We will show that no ψ_i with dimension 2 or more can occur in the decomposition of ψ.

Suppose ψ_1 has dimension $d \geqslant 2$, and $\psi_1 \cong \psi_2 \cong \cdots \cong \psi_s$ and there is no other ψ_i isomorphic to ψ_1 except for those ones. We may suppose $\psi_1 = \psi_2 = \cdots = \psi_s$.

After conjugation with an appropriate matrix in $\mathrm{GL}(n, \mathbb{C})$, we have

$$\psi(f) = \begin{pmatrix} \Psi_1(f) & 0 \\ 0 & \Psi'(f) \end{pmatrix}$$

for all $f \in F$, with

$$\Psi_1(f) = \begin{pmatrix} \psi_1(f) & 0 & \dots & 0 \\ 0 & \psi_1(f) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \psi_1(f) \end{pmatrix}$$

and in which Ψ' does not contain components isomorphic to ψ_1.

After conjugation with the same matrix in $\mathrm{GL}(n, \mathbb{C})$, Schur's lemma implies that A has the following shape:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

and we can write

$$A_1 = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ A_{21} & A_{22} & \dots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \dots & A_{ss} \end{pmatrix},$$

in which every A_{ij} is in $\mathbb{C}^{d \times d}$.

Since A commutes with $\psi(f)$ for every $f \in F$, it follows that $A_{ij}\psi_1(f) = \psi_1(f)A_{ij}$ for all $f \in F$ and all $1 \leqslant i, j \leqslant s$. Since ψ_1 is $\mathbb{C}$-irreducible, this implies $A_{ij} = \lambda_{ij}\mathbb{1}_d$, with $\lambda_{ij} \in \mathbb{C}$ for all $1 \leqslant i, j \leqslant s$. So we have

$$A_1 = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1s} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{s1} & \lambda_{s2} & \dots & \lambda_{ss} \end{pmatrix} \otimes \mathbb{1}_d = \Lambda \otimes \mathbb{1}_d$$

in which $\otimes$ denotes the Kronecker product of matrices (see e.g. [9, Section 10.4]). But then

$$\varphi_{A_1}(X) = \big(\varphi_\Lambda(X)\big)^d$$

and we see that $\varphi_{A_1}(X)$, and so also $\varphi_A(X)$, has roots of multiplicity at least d. This contradicts the previous lemma. □

Together with Theorem 2.3, we proved the following:

Corollary 3.3. *Let M be an infra-nilmanifold modeled on a free c-step nilpotent Lie group on n generators, with $n \leqslant 2c + 1$. Suppose M has holonomy group F and denote the associated abelianized rational holonomy representation with $\bar{\varphi} : F \to \mathrm{Aut}(\frac{N_\mathbb{Q}}{[N_\mathbb{Q}, N_\mathbb{Q}]})$. Then*

M admits an Anosov diffeomorphism

$$\Updownarrow$$

F is abelian, and each $\mathbb{Q}$-irreducible component $\bar{\varphi}_i$ of $\bar{\varphi}$ that occurs with multiplicity m, splits in more then $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.

In the last section of this paper we will show that this bound is sharp, by constructing an infra-nilmanifold modeled on a free c-step nilpotent Lie group on $n = 2c + 2$ generators and with a non-abelian holonomy group that admits an Anosov diffeomorphism.

4. Almost-Bieberbach groups with a given rational holonomy representation

For a torsion-free, finitely generated nilpotent group N, we write

$$\gamma_1(N) = N \quad \text{and} \quad \gamma_{n+1}(N) = \bigl[N, \gamma_n(N)\bigr]$$

to denote the lower central series of N and

$$\Gamma_i(N) = \sqrt{\gamma_i(N)} = \bigl\{x \in N \bigm| \exists n \in \mathbb{N}_0\colon\ x^n \in \gamma_i(N)\bigr\} = N \cap \gamma_i(N_{\mathbb{Q}}).$$

If there's no risk of confusion, we also write Γ_i in stead of $\Gamma_i(N)$.

In order to find new infra-nilmanifolds admitting an Anosov diffeomorphism, we want to be able to construct an almost-Bieberbach group E whose rational holonomy representation is induced by a given faithful representation $\varphi : F \to \operatorname{Aut}(N)$ of a finite group F into the group of automorphisms of a torsion-free, finitely generated nilpotent group N. The following theorem realizes this under certain condition.

Theorem 4.1. *Let $\varphi : F \hookrightarrow \operatorname{Aut}(N)$ be a faithful representation of a finite group F into the group of automorphisms of a torsion-free, finitely generated nilpotent group N, and denote with*

$$\bar{\varphi}_i : F \to \operatorname{Aut}\left(\frac{\Gamma_i}{\Gamma_{i+1}}\right) \cong \operatorname{Aut}\bigl(\mathbb{Z}^{k_i}\bigr)$$

the induced map. If there exists, for an $i \in \mathbb{N}_0$, a torsion-free extension

$$1 \to \frac{\Gamma_i}{\Gamma_{i+1}} \to \overline{E} \to F \to 1$$

inducing $\bar{\varphi}_i$, then there exists an almost-Bieberbach group E with holonomy group F, whose translation subgroup is a finite index subgroup of N and such that the rational holonomy representation $\psi : F \to \operatorname{Aut}(N_{\mathbb{Q}})$ coincides with $\varphi : F \to \operatorname{Aut}(N) \subseteq \operatorname{Aut}(N_{\mathbb{Q}})$.

Remark 4.2.

1. Notice that $\overline{E}$ is a Bieberbach group, since it is torsion-free and virtually abelian [4, Theorem 3.3.5]; but $\overline{E}$ does not necessarily have holonomy group F.
2. As the translation subgroup N' of E is claimed to be a finite index subgroup of N, we have that $N'_{\mathbb{Q}} = N_{\mathbb{Q}}$, and hence the rational holonomy representation of E does have its image in $\operatorname{Aut}(N_{\mathbb{Q}})$.

Proof. As any automorphism of N has a unique extension to an automorphism of the radicable hull $N_{\mathbb{Q}}$ of N, we will throughout this proof consider $\operatorname{Aut}(N)$ as being a subgroup of $\operatorname{Aut}(N_{\mathbb{Q}})$ and we will also view φ as being a representation into $\operatorname{Aut}(N_{\mathbb{Q}})$.

First suppose N is c-step nilpotent and $i = c$, so $\Gamma_{i+1} = 1$. Then we have a short exact sequence

$$1 \to \Gamma_i \to \overline{E} \to F \to 1$$

in which $\overline{E}$ is torsion-free. This short exact sequence induces an action of F on Γ_i, which is given by

$$\bar{\varphi}_i = \varphi_{|\Gamma_i} : F \to \operatorname{Aut}(\Gamma_i).$$

As $\bar{\varphi}(f)$, for all $f \in F$, can also be seen as an automorphism of $(\Gamma_i)_{\mathbb{Q}}$, there is an induced extension $\tilde{E}$ of F by $(\Gamma_i)_{\mathbb{Q}}$, which fits in the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_i \arrow[r] \arrow[d, tail] & \overline{E} \arrow[r] \arrow[d, tail] & F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & (\Gamma_i)_{\mathbb{Q}} \arrow[r] & \tilde{E} \arrow[r] & F \arrow[r] & 1.
\end{tikzcd}$$

As F is a finite group and $(\Gamma_i)_{\mathbb{Q}}$ is divisible, the bottom extension splits. Therefore, we can complete the above diagram as follows:

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_i \arrow[r] \arrow[d, tail] & \overline{E} \arrow[r] \arrow[d, tail] & F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & (\Gamma_i)_{\mathbb{Q}} \arrow[r] \arrow[d, equal] & \tilde{E} \arrow[r] \arrow[d, "\cong"] & F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & (\Gamma_i)_{\mathbb{Q}} \arrow[r] \arrow[d, equal] & (\Gamma_i)_{\mathbb{Q}} \rtimes F \arrow[r] \arrow[d, "\cong"] & F \arrow[r] \arrow[d, "\varphi", "\cong"'] & 1 \\
1 \arrow[r] & (\Gamma_i)_{\mathbb{Q}} \arrow[r] \arrow[d, tail] & (\Gamma_i)_{\mathbb{Q}} \rtimes \varphi(F) \arrow[r] \arrow[d, tail] & \varphi(F) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N_{\mathbb{Q}} \arrow[r] & N_{\mathbb{Q}} \rtimes \varphi(F) \arrow[r] & \varphi(F) \arrow[r] & 1.
\end{tikzcd}$$

It follows that we can view $\overline{E}$ as being a subgroup of $N_{\mathbb{Q}} \rtimes \varphi(F)$ in such a way that any element $\bar{e} \in \overline{E}$ can be written as a product

$$\bar{e} = z\alpha, \quad \text{with } z \in (\Gamma_i)_{\mathbb{Q}} \text{ and } \alpha \in \varphi(F).$$

Let m be the order of the finite group F, then for any element $\bar{e} = z\alpha \in \overline{E}$, we have that $\bar{e}^m = (z\alpha)^m \in \Gamma_i$.

Now, choose an element $z\alpha \in \overline{E}$ such that $z\alpha \notin \Gamma_i$ and write $z' = (z\alpha)^m$. We claim that z' is not the mth power of any element in Γ_i. Indeed suppose on the contrary that $z' = (z'')^m$ with $z'' \in \Gamma_i$. Then, since

$$(z\alpha)\big(z''\big)^m (z\alpha)^{-1} = (z\alpha)z'(z\alpha)^{-1} = (z\alpha)(z\alpha)^m(z\alpha)^{-1} = \big(z''\big)^m$$

and

$$\big((z\alpha)z''(z\alpha)^{-1}\big)^m = (z\alpha)\big(z''\big)^m (z\alpha)^{-1},$$

we have that

$$\big(z''\big)^m = \big((z\alpha)z''(z\alpha)^{-1}\big)^m.$$

Because Γ_i is a torsion-free nilpotent group, it follows that (see e.g. [10, Theorem 16.2.8])

$$z'' = (z\alpha)z''(z\alpha)^{-1}.$$

But then z'' and $z\alpha$ commute, so

$$\big(\big(z''\big)^{-1}(z\alpha)\big)^m = \big(z''\big)^{-m}(z\alpha)^m = \big(z'\big)^{-1}(z\alpha)^m = 1$$

and since $(z'')^{-1}(z\alpha) \in \overline{E}$, this contradicts the fact that $\overline{E}$ is torsion-free.

Remark that z' is also not an mth power of any other element of N, since $\frac{N}{\Gamma_i}$ is torsion-free.

Now consider the group $N^{m^i} = \mathrm{grp}\{g^{m^i} \mid g \in N\}$. This is a fully characteristic subgroup of N in which every element is the mth power of some element of N ([21, p. 113, Proposition 2]).

We now define E to be the group $E = N^{m^i} \cdot \overline{E}$ (in which N^{m^i} is normal). This group is torsion-free: suppose that there exists a torsion element

$$1 \neq x(z\alpha) \in E = N^{m^i} \cdot \overline{E} \subseteq N_{\mathbb{Q}} \rtimes \varphi(F) \quad \text{with } x \in N^{m^i} \text{ and } z\alpha \in \overline{E},$$

then $(x(z\alpha))^m = 1$ (every torsion element y in $N_{\mathbb{Q}} \rtimes \varphi(F)$ satisfies $y^m = 1$). On the other hand

$$\big(x(z\alpha)\big)^m = y(z\alpha)^m \quad \text{for some } y \in N^{m^i}.$$

So we must have that $y(z\alpha)^m = 1$, implying that $(z\alpha)^m \in N^{m^i}$. But then $(z\alpha)^m$ is the mth power of an element in N, and we already showed that this is impossible, unless $z\alpha = 1$. But this would mean that $1 \neq x$ is a torsion element in N^{m_i} which is impossible.

It is also clear that E, when seen now as a subgroup of $N_{\mathbb{R}} \rtimes \varphi(F)$, is discrete and cocompact, so E is an almost-Bieberbach group. Moreover, by construction, the associated rational holonomy representation of E equals $\varphi : F \to \mathrm{Aut}(N_{\mathbb{Q}})$.

To finish the proof, suppose the nilpotency class of N is larger then i. We can consider the nilpotent group $\frac{N}{\Gamma_{i+1}}$ and use the previous construction to find an almost-Bieberbach group $\overline{E}' \subseteq (\frac{N}{\Gamma_{i+1}})_{\mathbb{Q}} \rtimes \bar{\varphi}(F)$ (where $\bar{\varphi} : F \to \mathrm{Aut}(\frac{N}{\Gamma_{i+1}})$ is induced by φ). Note that $\bar{\varphi}$ is still injective.

Suppose $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ generate $\overline{E}'$, and choose $e_1, e_2, \ldots, e_n \in N_{\mathbb{Q}} \rtimes \varphi(F)$ such that e_i projects to $\bar{e}_i$. Then the group

$$E = \operatorname{grp}\bigl(\Gamma_{i+1} \cup \{e_1, e_2, \ldots, e_n\}\bigr)$$

is the almost-Bieberbach group we are looking for. $\square$

When constructing examples of infra-nilmanifolds with a given rational holonomy representation $\varphi : F \to \operatorname{Aut}(N_{\mathbb{Q}})$, induced by a representation $\varphi : F \to \operatorname{Aut}(N)$ as in the theorem above, it is often very useful when the induced representations

$$\varphi_i : F \to \operatorname{Aut}\left(\frac{\Gamma_i}{\Gamma_{i+1}}\right) \cong \operatorname{GL}(n_i, \mathbb{Z})$$

are totally reducible for all i. Here, we say that a representation $\varphi : F \to \operatorname{GL}(n, \mathbb{Z})$ is **totally reducible** if it splits into a direct sum of $\mathbb{Z}$-irreducible components.

The following lemma shows that this is always possible, if we allow to change the group N up to finite index (in which case $N_{\mathbb{Q}}$ and the induced representation $\varphi : F \to \operatorname{Aut}(N_{\mathbb{Q}})$ will not change).

Lemma 4.3. *Let N be a finitely generated torsion-free c-step nilpotent group and $\varphi : F \to \operatorname{Aut}(N_{\mathbb{Q}})$ a morphism, with F a finite group, such that $\varphi(f)(N) = N$ for all $f \in F$. Then there exists a finitely generated subgroup M of $N_{\mathbb{Q}}$ such that*

- *$\varphi(f)(M) = M$ for all $f \in F$,*
- *N is a subgroup of M of finite index, and*
- *for all i, the induced representation*

$$\bar{\varphi}_i : F \to \operatorname{Aut}\left(\frac{\sqrt{\gamma_i(M)}}{\sqrt{\gamma_{i+1}(M)}}\right) = \operatorname{GL}(n_i, \mathbb{Z})$$

is totally reducible.

Proof. We prove this lemma by induction on the nilpotency class. First suppose $c = 1$, then $N = \mathbb{Z}^n$ for an $n \in \mathbb{N}_0$.

We can choose a basis $\{b'_1, \ldots, b'_n\}$ of $\mathbb{Q}^n$ such that, with respect to this basis,

$$\varphi(f) = \begin{pmatrix} A_1(f) & 0 & \ldots & 0 \\ 0 & A_2(f) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_k(f) \end{pmatrix}$$

with $A_j : F \to \operatorname{GL}(m_j, \mathbb{Q})$ $\mathbb{Q}$-irreducible representations and with $A_j(f) \in \operatorname{GL}(m_j, \mathbb{Z})$ for all $f \in F$ and all j.

By changing the basis if necessary into $\{b_1, \ldots, b_n\} = \{\frac{b'_1}{k}, \ldots, \frac{b'_n}{k}\}$ for an appropriate $k \in \mathbb{N}_0$, we may suppose that $\mathbb{Z}^n$ is a subgroup of the group generated by $\{b_1, \ldots, b_n\}$. If we now take $M = \operatorname{grp}\{b_1, \ldots, b_n\}$, then M satisfies all the conditions.

Now suppose $c > 1$ and the lemma holds for nilpotency classes smaller then c. By induction, we know that we can find a subgroup $\overline{M}$ of $\frac{N_{\mathbb{Q}}}{\gamma_c(N_{\mathbb{Q}})}$ such that, if we denote by $\bar{\varphi} : F \to \frac{N_{\mathbb{Q}}}{\gamma_c(N_{\mathbb{Q}})}$ the induced representation on $\frac{N_{\mathbb{Q}}}{\gamma_c(N_{\mathbb{Q}})}$,

- $\bar{\varphi}(f)(\overline{M}) = \overline{M}$ for all $f \in F$,
- $\frac{N}{\sqrt{\gamma_c(N)}} = \frac{N}{\gamma_c(N_{\mathbb{Q}}) \cap N}$ is a subgroup of finite index in $\overline{M}$, and
- for all $i \in \{1, 2, \dots, c-1\}$, the induced representation

$$\bar{\varphi}_i : F \to \operatorname{Aut}\left(\frac{\sqrt{\gamma_i(\overline{M})}}{\sqrt{\gamma_{i+1}(\overline{M})}}\right)$$

  is totally reducible.

Now we can take a finitely generated subgroup M' of $N_{\mathbb{Q}}$ such that $\varphi(f)(M') = M'$, N is a subgroup of finite index in M' and the natural projection of M' in $\frac{N_{\mathbb{Q}}}{\gamma_c(N_{\mathbb{Q}})}$ coincides with $\overline{M}$: for example, if $\bar{m}_1, \dots, \bar{m}_r$ are generators of $\overline{M} \subseteq \frac{N_{\mathbb{Q}}}{\gamma_c(N_{\mathbb{Q}})}$, choose $m_1, \dots, m_r \in N_{\mathbb{Q}}$ such that m_i projects to $\bar{m}_i$, and let M' be the group generated by N and the $\varphi(f)(m_i)$ for all i and all $f \in F$.

First we have a look at the induced action on $\gamma_c(M')$. In the same way as in the case $c = 1$, we can choose an appropriate basis $\{b_1, \dots, b_n\}$ of $\sqrt{\gamma_c(N_{\mathbb{Q}})} \cong \mathbb{Q}^n$ such that $\sqrt{\gamma_c(M')}$ is a subgroup of $\operatorname{grp}\{b_1, \dots, b_n\}$, and moreover $\operatorname{grp}\{b_1, \dots, b_n\}$ is invariant under F and the induced action on $\operatorname{grp}\{b_1, \dots, b_n\}$ is totally reducible.

Now replace M' by $M = \operatorname{grp}(\{b_1, \dots, b_n\} \cup M')$. Then $\varphi(f)(M) = M$, N is a subgroup of M of finite index and the induced action $\bar{\varphi}_c$ on $\sqrt{\gamma_c(M)} = \operatorname{grp}\{b_1, \dots, b_n\}$ is totally reducible.

For $i < c$, we have

$$\frac{\sqrt{\gamma_i(M)}}{\sqrt{\gamma_{i+1}(M)}} = \frac{\sqrt{\gamma_i(M')}}{\sqrt{\gamma_{i+1}(M')}} = \frac{\sqrt{\gamma_i(\overline{M})}}{\sqrt{\gamma_{i+1}(\overline{M})}}$$

and therefore it follows that the induced representation $\bar{\varphi}_i$ is also totally reducible. □

5. Infra-nilmanifolds with a cyclic holonomy group

To be able to construct examples of infra-nilmanifolds with cyclic holonomy groups admitting an Anosov diffeomorphism using Theorem 4.1, we prove the following proposition.

Proposition 5.1. *Let $\varphi : \mathbb{Z}_n = \langle t \rangle \to \mathrm{GL}(k, \mathbb{Z})$ be a representation such that $\varphi(t)$ has 1 as an eigenvalue and is totally reducible. Then there is a torsion-free extension*

$$0 \longrightarrow \mathbb{Z}^k \longrightarrow E \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

that induces φ.

Proof. Since 1 is an eigenvalue of $\varphi(t)$, we can choose generators $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ of $\mathbb{Z}^k$ such that

$$\varphi(t)(\alpha_k) = \alpha_k \quad \text{and} \quad \text{then also } \varphi\bigl(t^m\bigr)(\alpha_k) = \alpha_k \quad \text{for all } m$$

and

$$\varphi(t)(\alpha_i) \in \operatorname{grp}\{\alpha_1, \dots, \alpha_{k-1}\} \quad \text{for all } i \in \{1, 2, \dots, k-1\}.$$

So we have

$$\mathbb{Z}^k = \{z_1\alpha_1 + \dots + z_k\alpha_k \mid z_i \in \mathbb{Z}\}$$

and

$$\mathbb{Q}^k = \{q_1\alpha_1 + \dots + q_k\alpha_k \mid q_i \in \mathbb{Q}\}.$$

Now take

$$\alpha = \left(\frac{1}{n}\alpha_k, \varphi(t)\right) \in \mathbb{Q}^k \rtimes \varphi(\mathbb{Z}_n)$$

and consider

$$E = \operatorname{grp}\{\alpha_1, \alpha_2, \dots, \alpha_k, \alpha\} \subseteq \mathbb{Q}^k \rtimes \varphi(\mathbb{Z}_n).$$

By calculating $\alpha\alpha_i\alpha^{-1}$ in E, we see that

$$\mathbb{Z}^k = \operatorname{grp}\{\alpha_1, \alpha_2, \dots, \alpha_k\} \lhd E$$

and

$$E/\mathbb{Z}^k \cong \mathbb{Z}_n$$

since $E/\mathbb{Z}^k$ is generated by $\bar{\alpha}$ and $\bar{\alpha}$ has order n in $E/\mathbb{Z}^k$. So E fits in a short exact sequence

$$0 \longrightarrow \mathbb{Z}^k \longrightarrow E \longrightarrow \mathbb{Z}_n \longrightarrow 0.$$

Moreover, in the same way we see that

$$\mathbb{Z}^{k-1} = \{z_1\alpha_1 + \dots + z_{k-1}\alpha_{k-1} \mid z_i \in \mathbb{Z}\} \lhd E$$

and

$$E/\mathbb{Z}^{k-1} \cong E' = \operatorname{grp}\{\bar{\alpha}_k, \bar{\alpha}\}.$$

Now because $\bar{\alpha}^n = \bar{\alpha}_k$, we have

$$E' = \operatorname{grp}\{\bar{\alpha}\} \cong \mathbb{Z}.$$

All together, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}^{k-1} \longrightarrow E \longrightarrow \mathbb{Z} \longrightarrow 0.$$

So E is torsion-free. $\square$

In the following proposition, we show that almost all faithful and rational representations of a cyclic group can be realized as the abelianized holonomy representation of an infra-nilmanifold modeled on a free c-step nilpotent Lie group.

Proposition 5.2. *Let $T : \mathbb{Z}_n = \langle t \rangle \to \mathrm{GL}(k, \mathbb{Z})$ $(k \geqslant 2)$ be a faithful representation and let $T_{\mathbb{Q}} : \mathbb{Z}_n \to \mathrm{GL}(k, \mathbb{Q})$ be the associated rational representation, then*

1. *There exists an n-dimensional flat manifold whose rational holonomy representation equals $T_{\mathbb{Q}}$ if and only if $T(t)$ has 1 as an eigenvalue.*
2. *For any $c \geqslant 2$, there exists an infra-nilmanifold modeled on the free c-step nilpotent Lie group N on k generators for which the associated abelianized rational holonomy representation equals $T_{\mathbb{Q}}$.*

Proof. Let us first consider the case of flat manifolds.

It is well known that if M is a flat manifold with holonomy group F and with rational holonomy representation $T_{\mathbb{Q}} : F \to \mathrm{GL}(k, \mathbb{Q})$, then $T_{\mathbb{Q}}(f)$ has 1 as an eigenvalue for any $f \in F$. So the condition that $T(t)$ has 1 as an eigenvalue is a necessary condition. On the other hand, Lemma 4.3 and Proposition 5.1 show that this condition is also sufficient.

Now we suppose that $c \geqslant 2$. Let Γ be the free c-step nilpotent group on k generators, then any representation $T : \mathbb{Z}_n \to \mathrm{GL}(k, \mathbb{Z})$ lifts to a representation $\tilde{T} : \mathbb{Z}_n \to \mathrm{Aut}(\Gamma)$, for which the induced representation on $\Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}^k$ equals T. We claim that we can always find eigenvalues λ_1, λ_2 of $T(t)$ such that $\lambda_1 = 1$ or $\lambda_1\lambda_2 = 1$. Indeed, since $T(t)$ has a characteristic polynomial in $\mathbb{Z}[X]$ and the eigenvalues of $T(t)$ are nth roots of unity, we see that, if $T(t)$ has a complex (non-real) eigenvalue λ, then $\bar{\lambda}$ is also an eigenvalue of M and $\lambda\bar{\lambda} = 1$. If M has only real eigenvalues, then those are either 1 or -1. If 1 is not an eigenvalue of M, then -1 occurs at least twice and $(-1)(-1) = 1$.

By Lemma 4.3 and Proposition 5.1, we see that the representation $\tilde{T} : \mathbb{Z}_n \to \mathrm{Aut}(\Gamma)$ satisfies the conditions of Theorem 4.1 (for $i = 1$ if $\lambda_1 = 1$ and $i = 2$ if $\lambda_1\lambda_2 = 1$), which guarantees the existence of the infra-nilmanifold we are looking for. $\square$

Combining the above result with Theorem 2.4, makes it now possible to give a full characterization of those representations $T : \mathbb{Z}_n \to \mathrm{GL}(k, \mathbb{Q})$ of finite cyclic groups, which occur as the abelianized rational holonomy representation of an infra-nilmanifold modeled on a free nilpotent Lie group admitting an Anosov diffeomorphism:

Theorem 5.3. *Let $k, c \geqslant 2$.*

A faithful representation $T : \mathbb{Z}_n \to \mathrm{GL}(k, Z) \hookrightarrow \mathrm{GL}(k, \mathbb{Q})$ can be realized as the rational holonomy representation of an infra-nilmanifold modeled on a free c-step nilpotent Lie group admitting an Anosov diffeomorphism if and only if each $\mathbb{Q}$-irreducible component of T that occurs with multiplicity m splits in more then $\frac{c}{m}$ components over $\mathbb{R}$.

Using the above, it is now easy to check when a cyclic group $\mathbb{Z}_n$ can occur as the holonomy group of an infra-nilmanifold modeled on a free c-step nilpotent Lie group on k generators, admitting an Anosov diffeomorphism. Example computations for $k \leqslant 20$ can be found in [23].

6. Examples of infra-nilmanifolds with a non-abelian holonomy group admitting Anosov diffeomorphisms

In this final section, we will use Theorem 4.1 to construct examples of infra-nilmanifolds having a non-abelian holonomy group and admitting an Anosov diffeomorphism. Therefore, we will need to work with Bieberbach groups, which are subgroups of $\mathbb{R}^n \rtimes \mathrm{GL}(n,\mathbb{R}) = \mathrm{Aff}(\mathbb{R}^n)$, the group of affine motions of $\mathbb{R}^n$. It is often useful to view $\mathrm{Aff}(\mathbb{R}^n)$ as a subgroup of $\mathrm{GL}(n+1,\mathbb{R})$. One can do this by mapping the affine map with linear part $A \in \mathrm{GL}(n,\mathbb{R})$ and translational part $a \in \mathbb{R}^n$ to the matrix $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$, where the 0 indicates a row of n zeroes. It follows that any n-dimensional Bieberbach group can be seen as a subgroup of $\mathrm{GL}(n+1,\mathbb{R})$.

There exists a Bieberbach group in dimension 3, generated by e_1, e_2, e_3, α and β, where these elements can be represented as 4×4-matrices as follows:

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(see e.g. [1]). This group fits in a short exact sequence

$$0 \to \mathbb{Z}^3 = \langle e_1, e_2, e_3 \rangle \to E \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0.$$

Consider the normal subgroup $N = \langle e_1^2, e_2, e_3 \rangle$ of E. This normal subgroup is isomorphic to $\mathbb{Z}^3$ and when we use a and b to denote the canonical projections of α and β, we find that

$$E/N = \langle a, b \mid a^4 = 1,\ b^2 = 1,\ ab = ba^3 \rangle \cong \mathcal{D}_4, \text{ the dihedral group of order 8.}$$

By calculating $\alpha e_i \alpha^{-1}$ and $\beta e_i \beta^{-1}$ for $i = 1, 2, 3$, we see that the induced action

$$\varphi_1 : \mathcal{D}_4 \to \mathrm{Aut}(N) \cong \mathrm{GL}(3,\mathbb{Z})$$

of $\mathcal{D}_4$ on N is given by the matrices

$$\varphi_1(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \varphi_1(b) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We just showed that there is a short exact sequence

$$1 \longrightarrow \mathbb{Z}^3 \longrightarrow E \longrightarrow \mathcal{D}_4 \longrightarrow 1,$$

with E torsion-free, such that this short exact sequence induces the (non-faithful) morphism $\varphi_1 : \mathcal{D}_4 \to \mathrm{GL}(3, \mathbb{Z})$.

More generally, we find the following corollary, which we will need to construct an almost-Bieberbach group with holonomy group $\mathcal{D}_4$.

Corollary 6.1. *Let*

$$\varphi = \varphi_1 \oplus \varphi_2 : \mathcal{D}_4 \to \mathrm{GL}(3+n, \mathbb{Z})$$

be a morphism with φ_1 as above and any choice of morphism $\varphi_2 : \mathcal{D}_4 \to \mathrm{GL}(n, \mathbb{Z})$, then there is a torsion-free extension

$$0 \longrightarrow \mathbb{Z}^{3+n} \longrightarrow E \longrightarrow \mathcal{D}_4 \longrightarrow 1$$

that induces φ.

Now consider the representation

$$\psi : \mathcal{D}_4 \to \mathrm{GL}(6, \mathbb{Z}): \quad a \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$b \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

Let N be the free 2-step nilpotent group on 6 generators, say $\{e_1, e_2, e_3, e_4, e_5, e_6\}$, and let $\tilde{\psi} : \mathcal{D}_4 \to \mathrm{Aut}(N)$ denote the morphism induced by ψ. So we have $\tilde{\psi}(a)(e_1) = e_2$, $\tilde{\psi}(a)(e_2) = e_1^{-1}$, etc.

The generators of $[N, N] = \sqrt{[N, N]}$ are exactly the $[e_i, e_j]$ with $1 \leqslant i < j \leqslant 6$. Therefore, define

$$\begin{array}{lllll} f_1 = [e_1, e_2] & f_4 = [e_1, e_3] & f_7 = [e_2, e_6] & f_{10} = [e_1, e_4] & f_{13} = [e_2, e_5] \\ f_2 = [e_3, e_4] & f_5 = [e_2, e_4] & f_8 = [e_3, e_5] & f_{11} = [e_2, e_3] & f_{14} = [e_3, e_6] \\ f_3 = [e_5, e_6] & f_6 = [e_1, e_5] & f_9 = [e_4, e_6] & f_{12} = [e_1, e_6] & f_{15} = [e_4, e_5]. \end{array}$$

Then, with respect to this basis $\{f_1, f_2, \dots, f_{15}\}$ of $[N, N]$, one can compute that $\tilde{\psi}_{|[N,N]}$ has a matrix representation

$$\tilde{\psi}(a) = \begin{pmatrix} \mathbb{1}_3 & & & & & & \\ & A_1 & & & & & \\ & & A_1 & & & & \\ & & & A_1 & & & \\ & & & & A_2 & & \\ & & & & & A_2 & \\ & & & & & & A_2 \end{pmatrix} \quad \text{with } A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

and

$$\tilde{\psi}(b) = \begin{pmatrix} -\mathbb{1}_3 & & \\ & \mathbb{1}_6 & \\ & & -\mathbb{1}_6 \end{pmatrix}.$$

By choosing an appropriate group M as in Lemma 4.3, we may suppose that, with respect to an appropriate basis of $\sqrt{[M, M]}$, the representation has the following form:

$$\tilde{\psi}(a) = \begin{pmatrix} \mathbb{1}_3 & & & & & & \\ & A & & & & & \\ & & A & & & & \\ & & & A & & & \\ & & & & A & & \\ & & & & & A & \\ & & & & & & A \end{pmatrix} \quad \text{with } A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\tilde{\psi}(b) = \begin{pmatrix} -\mathbb{1}_3 & & \\ & \mathbb{1}_6 & \\ & & -\mathbb{1}_6 \end{pmatrix}.$$

We see that

$$\tilde{\psi}_{|[M,M]} = \varphi_1 \oplus \varphi_2$$

with φ_1 as in Corollary 6.1. It follows that there is a torsion-free extension

$$1 \longrightarrow \sqrt{[M, M]} \longrightarrow E \longrightarrow \mathcal{D}_4 \longrightarrow 1$$

that induces $\tilde{\psi}$.

Using Theorem 4.1 we can conclude that there exists an almost-Bieberbach group E (modeled on the free 2-step nilpotent group with 6 generators) with abelianized rational holonomy representation $\psi : \mathcal{D}_4 \to \mathrm{GL}(6, \mathbb{Q})$ as given in (2).

The corresponding infra-nilmanifold has a non-abelian holonomy group, but does admit an Anosov diffeomorphism, since

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \otimes \mathbb{1}_2$$

commutes with $\psi(a)$ and $\psi(b)$.

One can easily see that this example can be generalized to infra-nilmanifolds modeled on a free c-step nilpotent Lie group on $2c+2$ generators, for all $c \geqslant 2$, such that the infra-nilmanifold has holonomy group $\mathcal{D}_4$ and admits an Anosov diffeomorphism. This shows that the bound we obtained in Corollary 3.3 is sharp.

References

[1] CARAT – Crystallographic AlgoRithms And Tables, Lehrstuhl B für Mathematik, Aachen, 2006, http://wwwb.math.rwth-aachen.de/carat/.

[2] S. Dani, Nilmanifolds with Anosov automorphisms, J. Lond. Math. Soc. (2) 18 (1978) 553–559.

[3] S.G. Dani, M.G. Mainkar, Anosov automorphisms on compact nilmanifolds associated with graphs, Trans. Amer. Math. Soc. 357 (6) (2005) 2235–2251.

[4] K. Dekimpe, Almost-Bieberbach Groups: Affine and Polynomial Structures, Lecture Notes in Math., vol. 1639, Springer-Verlag, 1996.

[5] K. Dekimpe, S. Deschamps, Anosov diffeomorphisms on a class of 2-step nilmanifolds, Glasg. Math. J. 45 (2003) 2269–2280.

[6] K. Dekimpe, P. Igodt, S. Kim, K.B. Lee, Affine structures for closed 3-dimensional manifolds with NIL-geometry, Quart. J. Math. Oxford (2) 46 (1995) 141–167.

[7] K. Dekimpe, K. Verheyen, Anosov diffeomorphisms on infra-nilmanifolds modeled on a free 2-step nilpotent Lie group, Groups Geom. Dyn. 3 (4) (2009) 555–578.

[8] K. Dekimpe, K. Verheyen, Anosov diffeomorphisms on nilmanifolds modeled on a free nilpotent Lie group, Dyn. Syst. 24 (1) (2009) 121–177.

[9] L. Hogben (Ed.), Handbook of Linear Algebra Discrete Mathematics and Its Applications (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2007, Associate editors: Richard Brualdi, Anne Greenbaum and Roy Mathias.

[10] M. Kargapolov, J. Merzljakov, Fundamentals of the Theory of Groups, Grad. Texts in Math., vol. 62, Springer-Verlag, 1979.

[11] J. Lauret, Examples of Anosov diffeomorphisms, J. Algebra 262 (1) (2003) 201–209.

[12] J. Lauret, C.E. Will, Nilmanifolds of dimension $\leqslant 8$ admitting Anosov diffeomorphisms, Trans. Amer. Math. Soc. 361 (5) (2009) 2377–2395.

[13] K.B. Lee, There are only finitely many infra-nilmanifolds under each nilmanifold, Quart. J. Math. Oxford (2) 39 (1988) 61–66.

[14] M.G. Mainkar, Anosov automorphisms on certain classes of nilmanifolds, Glasg. Math. J. 48 (1) (2006) 161–170.

[15] M.G. Mainkar, C.E. Will, Examples of Anosov Lie algebras, Discrete Contin. Dyn. Syst. 18 (1) (2007) 39–52.

[16] W. Malfait, Anosov diffeomorphisms on nilmanifolds of dimension at most six, Geom. Dedicata 79 (3) (2000) 291–298.

[17] A. Manning, There are no new Anosov diffeomorphisms on tori, Amer. J. Math. 96 (3) (1974) 422–429.

[18] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of Number Fields, Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences), vol. 323, Springer-Verlag, Berlin, 2000.

[19] T.L. Payne, Anosov automorphisms of nilpotent Lie algebras, J. Mod. Dyn. 3 (1) (2009) 121–158.

[20] H.L. Porteous, Anosov diffeomorphisms of flat manifolds, Topology 11 (1972) 307–315.

[21] D. Segal, Polycyclic Groups, Cambridge University Press, 1983.

[22] I. Stewart, D. Tall, Algebraic Number Theory, second ed., Chapman and Hall Math. Ser., Chapman and Hall, London, 1987.

[23] K. Verheyen, Anosov diffeomorphisms on infra-nilmanifolds modeled on a free nilpotent Lie group, PhD thesis, KULeuven, 2009.